Supporting Information for M. Rudner *et al.* "Generating Entanglement and Squeezed States of Nuclear Spins in Quantum Dots"

I. SQUEEZING OF THE WIGNER DISTRIBUTION

In this section, we provide a mathematical description of squeezing by analyzing the evolution of a nuclear spin state characterized by a Gaussian Wigner distribution. As discussed in the main text, for a large spin initially oriented in the x direction, and for short times before the Wigner distribution extends significantly around the Bloch sphere, the Wigner distribution in a locally flat patch of Bloch sphere evolves as $f_t(I_y, I_z) = \mathcal{A}e^{-\frac{1}{2}\mathbf{v}^T Q \mathbf{v}}$, see Eq.(10), with

$$\mathbf{v} = \begin{pmatrix} I^y \\ I^z \end{pmatrix}, \quad Q = \frac{1}{\Delta I^2}\begin{pmatrix} 1 & \lambda I t \\ \lambda I t & 1 + (\lambda I t)^2 \end{pmatrix}. \tag{S1}$$

Here $\Delta I = \Delta I_0^{y,z}$ characterizes the transverse fluctuations in the initial nuclear spin state.

For times $t > 0$, the circular Wigner distribution is deformed to an ellipse, with major and minor axes determined by the quadratic form Q in Eq.(S1). As shown in Fig.3 of the main text, stretching in one direction (y') is accompanied by squeezing in the perpendicular direction (z'), such that the phase space volume of the Wigner distribution is preserved. The major and minor axes y' and z', which lie parallel to the eigenvectors of Q, are rotated relative to y and z by an angle θ:

$$\begin{pmatrix} I^{y'} \\ I^{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} I^y \\ I^z \end{pmatrix}. \tag{S2}$$

The angle θ can be found by extremizing the quantity

$$W = \mathbf{w}_\theta^T Q \mathbf{w}_\theta, \quad \mathbf{w}_\theta = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} \tag{S3}$$

with respect to θ. Using the identity $[1 - \tan^2\theta]/2\tan\theta = \cot 2\theta$, we find

$$\cot 2\theta = \lambda I t/2. \tag{S4}$$

Note that Eq.(S4) has two solutions $\theta_{1,2}$ separated by $90°$, as expected for a symmetric form.

In the eigenbasis, we write

$$f_t(I^{y'}, I^{z'}) = \mathcal{A}\exp\left[-\frac{1}{2}\left(\frac{I^{y'}}{\Delta I_+(t)}\right)^2 - \frac{1}{2}\left(\frac{I^{z'}}{\Delta I_-(t)}\right)^2\right], \tag{S5}$$

with

$$\Delta I_{\pm}^2(t) = \Delta I^2 \left(1 + \frac{(\lambda I t)^2}{2}\left[1 \mp \sqrt{1 + \frac{4}{(\lambda I t)^2}}\right]\right)^{-1}. \quad \text{(S6)}$$

In the long time limit $\lambda I t \gg 1$, the width $\widetilde{\Delta I}(t) \equiv \Delta I_-(t)$ of the squeezed component reduces to Eq.(11).

II. PHASE DIFFUSION

The effect of time-dependent fluctuations of electron spin polarization about the mean field value can be analyzed within the rate equation model by introducing a time-dependent quantity

$$\tilde{S}^z(t) = S^z + \delta S^z(t). \quad \text{(S7)}$$

The fluctuating part δS^z can be modeled as delta-correlated noise $\langle \delta S^z(t')\delta S^z(t'')\rangle \propto \delta(t' - t'')$, with an intensity determined by the rate process, Eq.(5). As shown in Supplementary Section III, such noise generates phase diffusion,

$$\langle \delta\theta^2(t)\rangle = \kappa t, \quad \kappa = 2A^2 \frac{(W + \Gamma_1)W}{(2W + \Gamma_1)^3}, \quad \text{(S8)}$$

where $\delta\theta$ is the fluctuating part of the Larmor precession angle, $I^x + iI^y \propto e^{i(\theta+\delta\theta)}$. The phase diffusion can be accounted for by adding a diffusion term with diffusivity $\tilde{\kappa} = I^2\kappa$ to the equation describing the time evolution of the Wigner distribution.

An important consequence of phase diffusion is non-conservation of phase volume, which can be illustrated by the evolution of a Gaussian Wigner distribution. Similar to the mean-field case, Eq.(10), such a distribution evolves in time as

$$f_t(I^y, I^z) = \mathcal{A}'(t) \exp\left[-\frac{(I^z)^2}{2\Delta I^2} - \frac{(I^y + I\lambda t I^z)^2}{2(\Delta I^2 + \tilde{\kappa} t)}\right], \quad t > 0. \quad \text{(S9)}$$

Initially, phase diffusion leads to a broadening of the Wigner distribution, characterized by the factor $\sqrt{1 + \tilde{\kappa}t/\Delta I^2}$, which grows like $t^{1/2}$ for $\tilde{\kappa}t > \Delta I^2$. At later times, $\Delta I \lambda t I \gtrsim \sqrt{\tilde{\kappa} t}$, the behavior is dominated by the linear in t twisting/stretching dynamics. Therefore for times satisfying $t > t_{\rm noise} = \frac{2A^2\kappa}{N\lambda^2}$, the coherent stretching overwhelms the effect of phase diffusion.

The efficiency of squeezing in the presence of phase diffusion can be estimated as follows. At long times $t \gg t_{\rm noise}$, the factor $\sqrt{1 + \tilde{\kappa}t/\Delta I^2}$ describes an increase of the width of the

Wigner distribution compared to its ideal squeezed value $\widetilde{\Delta I}$ in Eq.(11). Combining the $t^{1/2}$ smearing due to phase diffusion with the t^{-1} squeezing, we find that the width of the Wigner distribution decreases as $t^{-1/2}$ at long times:

$$\widetilde{\Delta I}_{\rm noise} = \widetilde{\Delta I}(t)\sqrt{1+\tilde{\kappa}t/\Delta I^2} \approx \frac{\tilde{\kappa}^{1/2}}{\lambda I}t^{-1/2} \tag{S10}$$

This expression describes the slowing of squeezing due to phase diffusion.

III. CALCULATION OF THE PHASE DIFFUSION CONSTANT

To analyze phase diffusion, we need to calculate the generating function for spin fluctuations driven by up-down and down-up switching. Denoting the two switching rates as W and W', we can obtain the generating function for spin fluctuations during the time interval $0 < t' < t$ by approximating a continuous Poisson process by a discrete Markov process with a small time step $\Delta \ll W^{-1}, (W')^{-1}, t$. We have

$$\chi(\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\rm T} \left[e^{i\Delta(\lambda/2)\sigma_3} R_\Delta\right]^N \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \tag{S11}$$

$$R_\Delta = \begin{pmatrix} 1-W\Delta & W\Delta \\ W'\Delta & 1-W'\Delta \end{pmatrix}, \quad N = \frac{t}{\Delta},$$

where $W' = W + \Gamma_1$. Taking the limit $\Delta \to 0$, $N \to \infty$ we obtain an expression

$$\chi(\lambda) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\rm T} e^M \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \tag{S12}$$

$$M = t\begin{pmatrix} i\lambda/2 - W & W \\ W' & -i\lambda/2 - W' \end{pmatrix}. \tag{S13}$$

The generating function (S12) provides a full description of the statistics of phase fluctuations, $\theta_t = \int_0^t S_Z(t)dt$, by encoding all its cumulants:

$$\ln\chi(\lambda) = \sum_{k=1}^{\infty} m_k \frac{(i\lambda)^k}{k!}, \tag{S14}$$

with m_1 and m_2 giving the expectation value $\langle S_z\rangle t$ and the variance $\langle(\theta_t - \langle\theta_t\rangle)^2\rangle$, respectively. The latter quantity yields the phase diffusion constant via $m_2 = \kappa t$.

Matrix exponential e^M can be evaluated by writing it in terms of Pauli matrices, $M = x_0 + x_i\sigma_i$, where

$$x_0 = -W_+, \quad x_1 = W_+, \quad x_2 = iW_-, \quad x_3 = i\lambda/2 - W_-, \tag{S15}$$

and we defined $W_\pm = (W \pm W')/2$. We have

$$e^M = e^{x_0 t}\left(\cosh(Xt) + \frac{\sinh(Xt)}{X}x_i\sigma_i\right) \tag{S16}$$

where $X^2 = x_1^2 + x_2^2 + x_3^2 = W_+^2 - \lambda^2/4 - i\lambda W_-$. Plugging this expression for e^M in Eq.(S12), we find

$$\chi(\lambda) = 2e^{x_0 t}\left(\cosh(Xt) + \frac{\sinh(Xt)}{X}x_1\right), \tag{S17}$$

an exact expression which is valid both at short times and at long times.

To analyze fluctuations in the steady state, we focus on the long times $t \gg W^{-1}, (W')^{-1}$. In this limit, the behavior of $\chi(\lambda)$ can be understood by replacing $\cosh Xt$ and $\sinh Xt$ by e^{Xt}, giving

$$\ln\chi(\lambda) \approx (X - W_+)t = -\frac{\lambda^2/4 + i\lambda W_-}{X + W_+}t \tag{S18}$$

Taylor expanding this expression up to order λ^2 we find the first and second cumulants of phase fluctuations:

$$\ln\chi(\lambda) = -i\lambda\frac{W_- t}{2W_+} + \frac{(i\lambda)^2}{2}\frac{(W_+^2 - W_-^2)t}{4W_+^3} + O(\lambda^3) \tag{S19}$$

Substituting $W' = W + \Gamma_1$, we obtain the time-averaged polarization and the phase diffusion constant

$$\langle S_z\rangle = \frac{1}{2}\frac{\Gamma_1}{2W + \Gamma_1}, \quad \kappa = 2\frac{(W + \Gamma_1)W}{(2W + \Gamma_1)^3} \tag{S20}$$

Crucially, the phase diffusion slows down when the switching rates W and W' grow, which justifies our motional averaging approximation.